6.1 Reciprocal, Quotient, and Pythagorean Identities

Warm-up

Write each expression with a common denominator. Determine the restrictions.

a) $\frac{a}{b}+\frac{c}{d}$

b) $\dfrac{\frac{a}{b}}{\frac{c}{d}}$

c) $\dfrac{\frac{a}{c}+\frac{1}{b}}{b+\frac{c}{a}}$

Definition

Trigonometric identity

The equation $\tan\theta\cos\theta=\sin\theta$ is identity because it is true for all values of θ **except** $\theta=\frac{\pi}{2}+\pi k$**,** where k is an integer (where $\tan\theta$ is not defined). You can prove this is true by graphing $y_1=\tan\theta\cos\theta$ and $y_2=\sin\theta$

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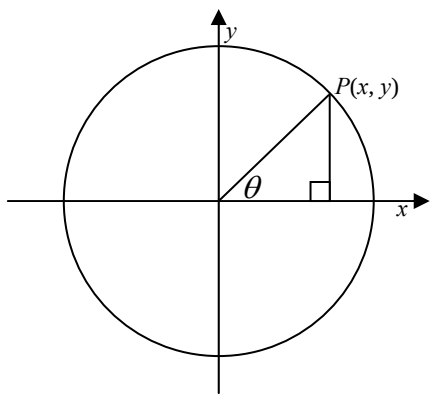
The most complete method for proving trigonometric identities uses algebra. This method can involve simplifying, factoring, and re-writing expressions.

In Chapter 4, six trigonometric functions were defined in terms of angle θ in standard position in a circle with a radius, r, and a terminal point P(x, y) on the circle.



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

<u>Some basic trigonometric identities (on formula sheet)</u>

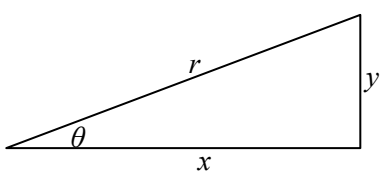
Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\cot\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities



$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$

$$\therefore \sin^2\theta = \frac{y^2}{r^2} \qquad \therefore \cos^2\theta = \frac{x^2}{r^2}$$

Pythagorean Theorem tells us that:

$$y^2 + x^2 = r^2 \qquad \text{or} \qquad y^2 + x^2 = r^2 \qquad \text{or} \qquad y^2 + x^2 = r^2$$

$$\frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{r^2}{r^2} \qquad \frac{y^2}{y^2} + \frac{x^2}{y^2} = \frac{r^2}{y^2} \qquad \frac{y^2}{x^2} + \frac{x^2}{x^2} = \frac{r^2}{x^2}$$

Note: Pythagorean Theorem is the only one of the identities that you can manipulate.

Identifying non-permissible values then Proving an Identity
A trigonometric expression, like an algebraic expression, cannot have a zero in the denominator.

Example 1: a) Determine the non-permissible values in degrees for the equation $\tan\theta\cos\theta = \sin\theta$.

b) Numerically verify that $\theta = 45°$ is a solution of the equation.

c) Prove algebraically that $\tan\theta\cos\theta = \sin\theta$

Example 2: For the expression: $\dfrac{\sin x + \tan x}{\cos x + 1} = \tan x$,

a) Identify the restriction on the variable.

b) Prove the identity.

$$\frac{\sin x + \tan x}{\cos x + 1} = \tan x$$

Simplifying Expressions

Example 3: Simplify the expression $\frac{1}{1+\cos x}+\frac{1}{1-\cos x}$

Several strategies that are often successful were used in this proof. Describe the strategies used.

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There is often more than one correct way to prove an identity. Sometimes it helps to work on **both** sides of the equation until they simplify to the same expression.
Based on experience the more complicated-looking side is the best place to start.

Example 4: Prove $\frac{\csc\theta-1}{\cot\theta}=\frac{\cot\theta}{\csc\theta+1}$. State any restrictions on θ.

Note:

- In this example the two sides appear symmetrical: there is no "harder" side to start on!
- Expressions such as $(\sin\theta-1)$ and $(\sin\theta+1)$ are called the **conjugates** of each other. Multiplying them sometimes produces a Pythagorean Identity: $(\sin\theta-1)(\sin\theta+1)$

$$\begin{aligned}&=\sin^2\theta+\sin\theta-\sin\theta-1\\&=\sin^2\theta-1\\&=\cos^2\theta\end{aligned}$$

Use this idea as a hint.

$$\frac{\csc\theta - 1}{\cot\theta} = \frac{\cot\theta}{\csc\theta + 1}$$

Tips

- The use of the conjugate in some proofs (see *Example 4*) is based on the principle of multiplying by 1. Do this to one side of the proof only.
- Once an identity is established, it **can be rearranged**. For example $\cot^2\theta = \csc^2\theta - 1$ is just another version of $\cot^2\theta + 1 = \csc^2\theta$, and any rearrangement can be used in future proofs.
- **Do not combine more than one step** in a proof on the same line. Your reasoning will not be clear and **you may be penalized**.
- If second degree terms are involved (ex. $\sin^2\theta$), consider using the Pythagorean Identities or factoring . **Avoid using square roots.**
- Reciprocal and Quotient Identities can be generalized; for example:

  $\csc^2\theta = \dfrac{1}{\sin^2\theta}$, $5\csc 2\theta = \dfrac{5}{\sin 2\theta}$, $3\cot 2\theta = \dfrac{3\cos 2\theta}{\sin 2\theta}$ *or* $3\cot 2\theta = \dfrac{3}{\tan 2\theta}$

- **Avoid common mistakes**, such as

  $\dfrac{\cos 2\theta}{2} \neq \cos\theta$, $\sin\theta + \cos\theta \neq 1$, $\sin 2\theta \neq 2\sin\theta \neq \sin^2\theta$.

You Try…

Prove $\dfrac{\sec\theta}{1-\cos\theta} = \dfrac{\sec\theta + 1}{\sin^2\theta}$. State any restrictions on θ.

Assignment: worksheet (do odd numbers)

6.2 Sum, Difference and Double-Angle Identities

Sum and Difference Identities	
$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$	$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$	$\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

Example 1: Express the following as a trigonometric function of a single angle: $\sin\pi\cos\frac{\pi}{5}-\cos\pi\sin\frac{\pi}{5}$.

Example 2: Consider the identity $\sin(90°-\theta)=\sin\theta$. Prove the identity algebraically.

Example 3: Find the exact value of $\cos\frac{7\pi}{12}$ without a calculator.

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Oh no! That means you will need to calculate the trig ratios for the following angles without a calculator

$\pm 15°$ or $\pm\frac{\pi}{12}$, $\pm 75°$ or $\pm\frac{5\pi}{12}$, $\pm 105°$ or $\pm\frac{7\pi}{12}$

For angles, $\pm 165°$ or $\pm\frac{11\pi}{12}$, $\pm 195°$ or $\pm\frac{13\pi}{12}$, …., you may have to add or subtract more than once.

Example 4: If $\sin A = \frac{2}{3}$ and $\cos B = -\frac{3}{5}$. Both $\angle A$ and $\angle B$ are in Quadrant 2, evaluate $\cos(A - B)$.

Assignment: worksheet (do 1-39 odd problems and 40-42)

6.2 Sum, Difference and Double-Angle Identities (continue…)

Double Angle Identities		
$\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $= 2\cos^2\theta - 1$ $= 1 - 2\sin^2\theta$	$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Example 1: If $\sin A = -\frac{1}{3}$ and $\angle A$ is in Quadrant 3, evaluate $\tan 2A$.

Example 2: Write $30\sin A\cos A$ as a single trigonometric ratio solution.

You try…Write each of the following as a single trigonometric ratio solution.

a) $\cos^2 5\theta - \sin^2 5\theta$

b) $3 - 6\sin^2 4\theta$

c) $\sin\theta\cos\theta$

Example 3: If $\sin A = \frac{2}{5}$ and $\angle A$ is in Quadrant 2, evaluate

a) $\cos 2A$

b) $\sin 2A$

Example 4: Prove the identity, $\dfrac{1-\cos 2x}{\sin 2x} = \tan x$.

$$\frac{1-\cos 2x}{\sin 2x} = \tan x$$

Assignment: worksheet

6.3 Proving Identities

Example 1: Prove the identity $\frac{\tan^2 x}{1+\sec x} = \frac{1-\cos x}{\cos x}$

$$\frac{\tan^2 x}{1+\sec x} = \frac{1-\cos x}{\cos x}$$

Example 2: Prove the identity $\dfrac{1}{1+\sin x} = \dfrac{\sec x - \sin x \sec x}{\cos x}$

$$\frac{1}{1+\sin x} = \frac{\sec x - \sin x \sec x}{\cos x}$$

Example#2: Prove the identity $\dfrac{\sin 2x - \cos x}{4\sin^2 x - 1} = \dfrac{\sin^2 x \cos x + \cos^2 x}{2\sin x + 1}$

$$\frac{\sin 2x - \cos x}{4\sin^2 x - 1} = \frac{\sin^2 x \cos x + \cos^2 x}{2\sin x + 1}$$

Recall: On page 5 of the notes, strategies when working with proofs.

Assignment: page 314 #1-4, 6, 7, 8, 10, 11