6.1 Reciprocal, Quotient, and Pythagorean Identities

Warm-up

Write each expression with a common denominator. Determine the restrictions.

a) \( \frac{a}{b} + \frac{c}{d} \)

b) \( \frac{a}{b} + \frac{c}{d} \)

c) \( \frac{a}{b} + \frac{1}{c} \)

Definition

Trigonometric identity

The equation \( \tan \theta \cos \theta = \sin \theta \) is identity because it is true for all values of \( \theta \) except \( \theta = \frac{\pi}{2} + \pi k \), where \( k \) is an integer (where \( \tan \theta \) is not defined). You can prove this is true by graphing \( y_1 = \tan \theta \cos \theta \) and \( y_2 = \sin \theta \).
The most complete method for proving trigonometric identities uses algebra. This method can involve simplifying, factoring, and re-writing expressions.

In Chapter 4, six trigonometric functions were defined in terms of angle $\theta$ in standard position in a circle with a radius, $r$, and a terminal point $P(x, y)$ on the circle.

Some basic trigonometric identities (on formula sheet)

**Reciprocal Identities**

\[
\csc \theta = \frac{1}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \cot \theta = \frac{1}{\tan \theta}
\]

**Quotient Identities**

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}
\]

**Pythagorean Identities**

\[
\sin \theta = \frac{y}{r} \quad \cos \theta = \frac{x}{r} \quad \therefore \sin^2 \theta = \frac{y^2}{r^2} \quad \therefore \cos^2 \theta = \frac{x^2}{r^2}
\]

Pythagorean Theorem tells us that:

\[
\begin{align*}
y^2 + x^2 &= r^2 \\
\frac{y^2}{r^2} + \frac{x^2}{r^2} &= \frac{r^2}{r^2} \\
\text{or} \quad \frac{y^2}{y^2} + \frac{x^2}{y^2} &= \frac{r^2}{y^2} \\
\text{or} \quad \frac{y^2}{x^2} + \frac{x^2}{x^2} &= \frac{r^2}{x^2}
\end{align*}
\]

Note: Pythagorean Theorem is the only one of the identities that you can manipulate.
Identifying non-permissible values then Proving an Identity
A trigonometric expression, like an algebraic expression, cannot have a zero in the denominator.

Example 1: a) Determine the non-permissible values in degrees for the equation \( \tan \theta \cos \theta = \sin \theta \).

b) Numerically verify that \( \theta = 45^\circ \) is a solution of the equation.

c) Prove algebraically that \( \tan \theta \cos \theta = \sin \theta \)
Example 2: For the expression: $\frac{\sin x + \tan x}{\cos x + 1} = \tan x$,

a) Identify the restriction on the variable.

b) Prove the identity.

$$\frac{\sin x + \tan x}{\cos x + 1} = \tan x$$
Simplifying Expressions

Example 3: Simplify the expression \[ \frac{1}{1 + \cos x} + \frac{1}{1 - \cos x} \]

Several strategies that are often successful were used in this proof. Describe the strategies used.

_______________________________________________________________________________________
_______________________________________________________________________________________
_______________________________________________________________________________________
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_______________________________________________________________________________________
_______________________________________________________________________________________

There is often more than one correct way to prove an identity. Sometimes it helps to work on both sides of the equation until they simplify to the same expression. Based on experience the more complicated-looking side is the best place to start.

Example 4: Prove \[ \csc \theta - 1 = \frac{\cot \theta}{\csc \theta + 1} \]. State any restrictions on \( \theta \).

Note:
- In this example the two sides appear symmetrical: there is no “harder” side to start on!
- Expressions such as \((\sin \theta - 1)\) and \((\sin \theta + 1)\) are called the conjugates of each other. Multiplying them sometimes produces a Pythagorean Identity: \((\sin \theta - 1)(\sin \theta + 1)\)
  \[
  = \sin^2 \theta + \sin \theta - \sin \theta - 1 \\
  = \sin^2 \theta - 1 \\
  = \cos^2 \theta
  \]
  Use this idea as a hint.
\[
\frac{\csc \theta - 1}{\cot \theta} = \frac{\cot \theta}{\csc \theta + 1}
\]

**Tips**

- The use of the conjugate in some proofs (see Example 4) is based on the principle of multiplying by 1. Do this to one side of the proof only.
- Once an identity is established, it **can be rearranged**. For example \( \cot^2 \theta = \csc^2 \theta - 1 \) is just another version of \( \cot^2 \theta + 1 = \csc^2 \theta \), and any rearrangement can be used in future proofs.
- **Do not combine more than one step** in a proof on the same line. Your reasoning will not be clear and **you may be penalized**.
- If second degree terms are involved (ex. \( \sin^2 \theta \)), consider using the Pythagorean Identities or factoring. **Avoid using square roots**.
- Reciprocal and Quotient Identities can be generalized; for example:
  \[
  \csc^2 \theta = \frac{1}{\sin^2 \theta}, \quad 5 \cot 2\theta = \frac{5}{\sin 2\theta}, \quad 3 \cot 2\theta = \frac{3 \cos 2\theta}{\sin 2\theta} \quad \text{or} \quad 3 \cot 2\theta = \frac{3}{\tan 2\theta}
  \]
- **Avoid common mistakes**, such as
  \[
  \frac{\cos 2\theta}{2} \neq \cos \theta, \quad \sin \theta + \cos \theta \neq 1, \quad \sin 2\theta \neq 2\sin \theta \neq \sin^2 \theta.
  \]
You Try...

Prove \( \frac{\sec \theta}{1 - \cos \theta} = \frac{\sec \theta + 1}{\sin^2 \theta} \). State any restrictions on \( \theta \).

Assignment: worksheet (do odd numbers)
### 6.2 Sum, Difference and Double-Angle Identities

#### Sum and Difference Identities

<table>
<thead>
<tr>
<th>Expression</th>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sin(\alpha + \beta))</td>
<td>(\sin\alpha \cos\beta + \cos\alpha \sin\beta)</td>
</tr>
<tr>
<td>(\sin(\alpha - \beta))</td>
<td>(\sin\alpha \cos\beta - \cos\alpha \sin\beta)</td>
</tr>
<tr>
<td>(\cos(\alpha + \beta))</td>
<td>(\cos\alpha \cos\beta - \sin\alpha \sin\beta)</td>
</tr>
<tr>
<td>(\cos(\alpha - \beta))</td>
<td>(\cos\alpha \cos\beta + \sin\alpha \sin\beta)</td>
</tr>
<tr>
<td>(\tan(\alpha + \beta))</td>
<td>(\tan\alpha + \tan\beta)</td>
</tr>
<tr>
<td>(\tan(\alpha - \beta))</td>
<td>(\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta})</td>
</tr>
</tbody>
</table>

**Example 1:** Express the following as a trigonometric function of a single angle: \(\sin\pi \cos\frac{\pi}{5} - \cos\pi \sin\frac{\pi}{5}\).

**Example 2:** Consider the identity \(\sin(90^\circ - \theta) = \sin\theta\). Prove the identity algebraically.
Example 3: Find the exact value of $\cos \frac{7\pi}{12}$ without a calculator.

Example 4: If $\sin A = \frac{2}{3}$ and $\cos B = -\frac{3}{5}$. Both $\angle A$ and $\angle B$ are in Quadrant 2, evaluate $\cos(A - B)$.

Assignment: worksheet (do 1-39 odd problems and 40-42)
### 6.2 Sum, Difference and Double-Angle Identities (continue…)

**Double Angle Identities**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin 2\theta = 2\sin \theta \cos \theta )</td>
<td>( \cos 2\theta = \cos^2 \theta - \sin^2 \theta )</td>
</tr>
<tr>
<td></td>
<td>( = 2\cos^2 \theta - 1 )</td>
</tr>
<tr>
<td></td>
<td>( = 1 - 2\sin^2 \theta )</td>
</tr>
<tr>
<td></td>
<td>( \tan 2\theta = \frac{2\tan \theta}{1 - \tan^2 \theta} )</td>
</tr>
</tbody>
</table>

**Example 1:** If \( \sin A = -\frac{1}{3} \) and \( \angle A \) is in Quadrant 3, evaluate \( \tan 2A \).

**Example 2:** Write \( 3\sin A \cos A \) as a single trigonometric ratio solution.

**You try…** Write each of the following as a single trigonometric ratio solution.

a) \( \cos^2 5\theta - \sin^2 5\theta \)

b) \( 3 - 6\sin^2 4\theta \)

c) \( \sin \theta \cos \theta \)
Example 3: If \( \sin A = \frac{2}{5} \) and \( \angle A \) is in Quadrant 2, evaluate

a) \( \cos 2A \)

b) \( \sin 2A \)
Example 4: Prove the identity, \( \frac{1 - \cos 2x}{\sin 2x} = \tan x \).

\[
\frac{1 - \cos 2x}{\sin 2x} = \tan x
\]
6.3 Proving Identities

Example 1: Prove the identity \( \frac{\tan^2 x}{1 + \sec x} = \frac{1 - \cos x}{\cos x} \)
Example 2: Prove the identity \[ \frac{1}{1 + \sin x} = \frac{\sec x - \sin x \sec x}{\cos x} \]
Example #2: Prove the identity \( \frac{\sin 2x - \cos x}{4 \sin^2 x - 1} = \frac{\sin^2 x \cos x + \cos^2 x}{2 \sin x + 1} \)
Recall: On page 5 of the notes, strategies when working with proofs.

Assignment: page 314 #1-4, 6, 7, 8, 10, 11